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APPLICATION OF NEW ALTERNATIVE MODELS OF FINITE ELEMENT METHOD IN PROBLEMS OF ROD TORSION

The work is dedicated to the problem solution of prismatic bars torsion with a rectangular section by the finite element method with the use of standard and alternative serendipian models. By the solving of the inverse problem considering the precise value for maximum shear stress, the new improved models on the biquadratic and bicubic serendipian elements were received. By using new alternative models as well as standard models, the shear stresses in two dangerous points of section and the torque for different ratio of the rectangle's sides were defined. The obtained results allow to solve various application problems of physical fields restoring occuedr in technical systems and objects by developing new mathematical models.

Keywords: *prismatic rod, stress function, torque, shear stresses, finite element, basic function.*

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ЗАСТОСУВАННЯ НОВИХ АЛЬТЕРНАТИВНИХ МОДЕЛЕЙ МЕТОДУ СКІНЧЕННИХ ЕЛЕМЕНТІВ У ЗАДАЧАХ ПРО КРУЧЕННЯ СТЕРЖНІВ

Роботу присвячено розв'язуванню задачі про кручення призматичних стержнів з прямокутним поперечним перерізом методом скінченних елементів з використанням стандартних та альтернативних серендипових моделей. Шляхом розв'язання оберненої задачі з урахуванням точного значення для максимального дотичного напруження було отримано нові вдосконалені моделі на біквадратичному та бікубічному серендипових елементах. З використанням нових альтернативних моделей, а також і стандартних моделей, було визначено дотичні напруження у двох небезпечних точках перерізу та крутний момент для різних відношень сторін прямокутника. Отримані результати дають можливість вирішувати різні прикладні проблеми, пов'язані з відновленням фізичних полів, що виникають в технічних системах та об'єктах, шляхом розробки нових математичних моделей.

Ключові слова: *призматичний стержень, функція напружень, крутний момент, дотичні напруження, скінчений елемент, базисна функція.*

Problem definition. Currently, approximate methods increasingly used in applied problems since the precise solution of differential equations with partial derivatives are quite bulky and it is very difficult or even impossible to find it. Therefore, for example, under bars torsion the approximate method of Ritz [1] is used (which gives the erroneously low result) or Treftz method [2] (which gives the exaggerated result) or two methods together to reduce the error of the approximate solution. But among many engineering problems one of the most popular computational methods is the finite element method (FEM) [3] that is successfully used in solving various problems. However, the problem of FEM schemes improvement in order to decrease the expenses on their implementation is not only relevant in Ukraine but also in the whole world.

Analysis of recent research and publications. The first works using FEM for structural mechanics problems are published in [4]. With the advent of computer engineering it became possible to expedite bulky calculations and to obtain more accurate results. However, work on improving of the universal computing applications continues. So in recent research [5], [6] due to geometric modelling, new improved FEM models used in this article were obtained.

Description of general problem unsolved aspects. The problem of torsion rectangular cross section bars using new improved models was considered in [6]. The torque and maximum shear stress were determined. Then these results were compared to the exact results obtained in [7] and the results for the standard models. But the value of the shear stress in another dangerous point was not determined; also research for rectangular aspect ratio of more than 2.5 was not conducted at all.

The purpose of the study. To implement the FEM to solve the torsion problem of prismatic rectangular cross section bars using standard and alternative models and to analyze the accuracy of the results.

Main research. This work is dedicated to solve torsion problems of prismatic bars with rectangular cross section $2a \times 2b$ and $b \geq a$ (fig. 1).

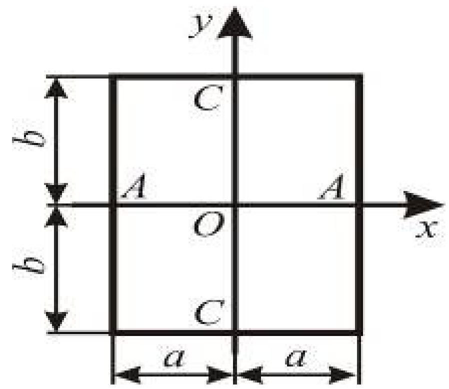


Figure 1 – The bar's cross section

The mathematical model of this problem is the Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\Theta \qquad (1)$$

where φ – the stress function (the Prandtl function) must satisfy the differential equation (1) and be equal to zero along the edge of the cross section;

G – the modulus of elasticity of the second kind (shear modulus),

Θ – the relative angle of twist per unit of length.

The way of solving the equation (1) using the membrane analogy [7] is shown in the theory of elasticity.

The torque, knowing the stress function, is determined by the formula

$$M_t = 2\int_{-a}^{a}\int_{-b}^{b}\varphi(x, y)dxdy, \tag{2}$$

where a, b – half sides of the rectangular section (fig. 1).

The exact calculation of M_t is quite bulky and is reproduced in [7]. For practical purposes the simplified formula for M_t is used [7]

$$M_t = k_1 G\Theta(2b)(2a)^3, \tag{3}$$

where k_1 – numerical coefficient that depends on the ratio of b/a.

It is also argued that the maximum shear stresses $\tau_{\max}$ in the torsion of rectangular cross section bar occur at the surface in the middle of longer sides of rectangular section (in points A) and are equal [7]

$$\tau_{max} = \frac{M_t}{k_2(2b)(2a)^2}, \tag{4}$$

where k_2 – numerical coefficient that depends on b/a.

The shear stresses occured at the surface in the middle of smaller sides of rectangle (in points C) are smaller. They are calculated by the formula

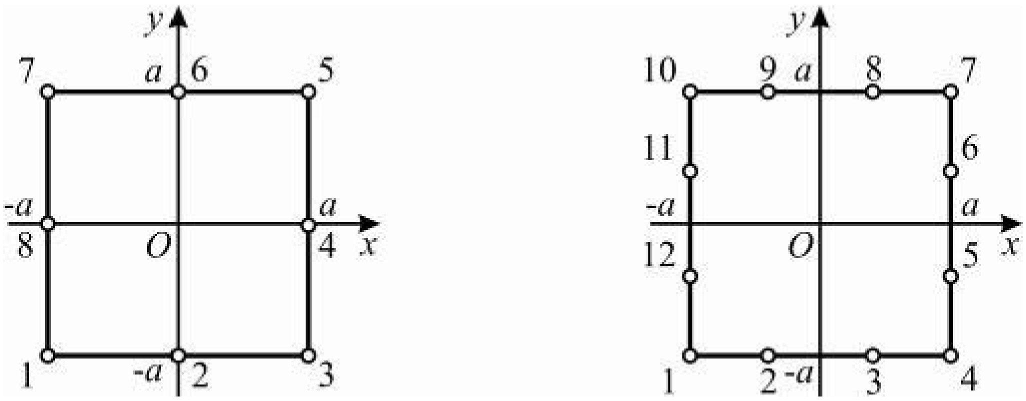
$$\tau = k\tau_{max}, \tag{5}$$

where $k = k_1/k_2$.

Some values of k_1, k_2, k coefficients are listed in the tabl00e in [7].

The same problem was being solved in [6]. A variety of the standard models were used. For the further study, two models from [6], which produced the most accurate results, are used.

Model 1. We are using biquadratic elements of the serendipian family (8 nods) (fig. 2, *a*) [8, 9], covering the bar's cross section with the four 8-nods elements



a) *b)*

Figure 2 – Biquadratic (*a*) and bicubic (*b*) serendipian elements

The basic functions are [4]

$$N_1^{(1)}(x,y)=\frac{1}{4}(1-x)(1-y)(-1-x-y),$$
$$N_2^{(1)}(x,y)=\frac{1}{2}(1-x^2)(1-y). \qquad (6)$$

Model 2. Bicubic serendipian elements (12 nods) (fig. 2, *b*) [8, 9], covering the bar's cross section with the four such elements are used. The basic functions are [4]

$$N_1^2(\xi,\eta)=\frac{1}{32}(1-\xi)(1-\eta)\left[-10+9(\xi^2+\eta^2)\right],$$
$$N_2^2(\xi,\eta)=\frac{9}{32}(1-\xi^2)(1-\eta)(1-3\xi). \qquad (7)$$

In addition to standard models (6), (7), due to geometric modelling, new alternative models on these elements are obtained [5], and the unlimited quantity of models with different features through "scaling" them is received. By the solution of the inverse problem with account of the precise meaning for $\tau_{\max}$ for the square cross section, the new models of the biquadratic element were received [6]

$$N_1^{(*)}(x,y)=\frac{1}{2000}(1-x)(1-y)(-569-569x-569y-69xy),$$
$$N_1^{(*)}(x,y)=\frac{1}{2000}(1-x^2)(1-y)(1069+69y). \qquad (8)$$

and on the bicubic element

$$N_1^{(**)}(\xi,\eta)=\frac{-1}{32000}(1-\xi)(1-\eta)\left[13789-12789(\xi^2+\eta^2)+3789\xi^2\eta^2\right],$$
$$N_2^{(**)}(\xi,\eta)=\frac{3}{64000}(1-\xi^2)(1-\eta)(7263-21789\xi+842\eta-842\xi\eta-421\eta^2+2947\xi\eta^2). \qquad (9)$$

These optimized models were tested for solving problems of torsion of prismatic rectangular cross section bars with ratio sides of $1\le b/a\le 2{,}5$ only. In this article the study is continued. The relative error of calculations for $\tau_{\max}$ and M_t in comparison with accurate [7] for rectangles with the ratio of sides $b/a>2{,}5$ is determined. These results are shown in the table 1. The comparative analysis with the corresponding results for the standard models (6) and (7) are shown in this table as well.

Table 1 – Relative errors for $\tau_{\max}$ and M_t for rectangular cross section bar, [%]

Model \ b/a	3,0		5,0		7,0		10,0	
	$\tau_{\max}$	M_t	$\tau_{\max}$	M_t	$\tau_{\max}$	M_t	$\tau_{\max}$	M_t
(6)	13,2	4,1	13,8	4,0	14.0	4,0	14,1	3,9
(7)	18,1	2,8	18,9	2,5	19,2	2,4	19,3	2,4
(8)	8,6	2,8	9,1	2.5	9,3	2,3	9,4	2,3
(9)	1,3	0,8	1,6	0,7	1,7	0,67	1,7	0,67

It has to be noted, that these results were obtained dividing the bar's cross section by only four elements. As shown in [6], covering the bar's square cross section with 16 elements significantly improves the results of calculation for standard models (6) and (7). The same situation occurs on rectangles with different sides ratio. The best result was obtained for the model (7).

However the determination of the shear stress τ in another dangerous point, which is located in the middle of the smaller side of rectangle (point C) using new improved models was not conducted in [6]. Note, that the determination of the value for τ in point C is necessary not only to get more results by different methods and compare them with accurate (although this is important), but also because it opens up the possibility to solve complex problems in which the bar of rectangular cross section works not only in torsion, but, for example, bending with torsion too. In this case it is difficult resistance. As it is known from the materials mechanics, the greatest normal stresses [10] occur in the points of smaller sides of the rectangle, thus in point C too. Therefore, when calculating bar structures for strength for the third or fourth failure theories the value of shear stress τ in point C can't be ignored because this value is also considerable. For the different ratios of rectangle's sides it is within $0{,}743\tau_{\max} \le \tau \le \tau_{\max}$ [7].

The obtained relative errors of calculations for τ in point C using new alternative models in comparison to precise [7] and standard models for different ratio of rectangle's sides b/a are shown in table 2.

Table 2 – Relative errors for τ in rectangular cross section bar, [%]

b/a / Model	1,0	2,0	3,0	5,0	7,0	10,0
	τ	τ	τ	τ	τ	τ
(6)	2,37	9,12	12,6	13,8	14,6	15,2
(7)	12,06	14,3	16,1	17,6	18,4	18,7
(8)	0	3,0	4,2	5,3	5,4	5,6
(9)	0	0,32	0,81	0,94	1,2	1,24

Conclusions. The new alternative models of the finite element method on the biquadratic and bicubic elements were received. Using these models and dividing the bar's cross section only by four elements the quite precise results were obtained. While using the standard models, the acceptable results were obtained only covering the bar's cross section with 16 elements. But the amount of computing work was essentially increased.

The obtained results allow to solve various application problems of restoring of the physical fields that occur in technical systems and objects by developing new mathematical models.

References

1. *Ascher U. M. Computer methods for ordinary differential equations and differential-algebraic equations / U. M. Ascher, L. R. Petzold. – SIAM, 1998.*
   ISBN 0898714125
2. *Numerical Recipes in C. Cambridge / William H. Press, Brian P. Flannery, Saul A. Teukolsky, William T. Vetterling. – Cambridge University Press, 1988.*
3. *Стренг Г. Теория метода конечных елементов / Г. Стренг, Дж. Фикс; перевод с англ. В. И. Агошков, В. А. Василенко, В. В. Шайдуров. – М. : Мир, 1977. – 349 с.*
   Streng G. The theory of the finite element method / G. Streng. – M. : Mir, 1977. – 349 s.
   ISBN10: 096140888X, ISBN13: 9780961408886
   http://pnu.edu.ru/media/filer_public/2013/04/10/6-15_streng_1977.pdf

4. *Зенкевич О. Конечные элементы и аппроксимации / О. Зенкевич, К. Морган. – М. : Наука, 1979. – 328 с.*
Zenkevich O. Konechnye elementy i approksimatsii / O. Zenkevich, K. Morgan. – M. : Nauka, 1979. – 328 s.
ISBN10: 0471982407, ISBN13: 9780471982401
http://www.rk5.msk.ru/Knigi/MKE/Zenkevich.pdf
5. *Камаєва С. О. Дослідження властивостей серендипових моделей / С. О. Камаєва // Актуальные достижения европейской науки – 2009: V Междунар. науч.-практ. конф. (17 – 25.06.2009) – Днепропетровск, 2009. – Т. 11. – С 55 – 58.*
Kamaeva S. O. Doslidzhennya vlastivostey serendipovih modeley / S. O. Kamaeva // Aktualnye dostizheniya evropeyskoy nauki – 2009: V Mezhdunar. nauch.-prakt. konf. (17 – 25.06.2009) – Dnepropetrovsk, 2009. – T. 11. – S 55 – 58.
http://www.rusnauka.com/18_EN_2009/Matemathics/48807.doc.htm
6. *Федоришин Н. . Нові удосконалені моделі методу скінченних елементів. / Н. Г. Федоришин, С. О. Камаєва // НАУКОВІ НОТАТКИ: міжвузівський збірник. – Луцьк: ЛНТУ, 2010. – Вип. 28. – С. 539 – 543.*
Fedorishin N. . Novi udoskonaleni modeli metodu skinchennih elementiv. / N. G. Fedorishin, S. O. Kamaeva // NAUKOVI NOTATKI: mizhvuzivskiy zbirnik. – Lutsk: LNTU, 2010. – Vip. 28. – S. 539 – 543.
http://nbuv.gov.ua/UJRN/Nn_2010_28_111
7. *Тимошенко С. П. Теория упругости / С. П. Тимошенко, Дж. Гудьер. – М. : Наука, 1979. – 560 с.*
Timoshenko S. P. Teoriya uprugosti / S. P. Timoshenko, Dzh. Guder. – M. : Nauka, 1979. – 560 s.
ISBN10: 0070642702, ISBN13: 9780070642706 https://goo.gl/ijHVGb
8. *Сегернлинд Л. Применение метода конечних элементов / Сегернлинд Л. – М.: Мир, 1979. – 392 с.*
Segernlind L. Primenenie metoda konechnih elementov / Segernlind L. – M.: Mir, 1979. – 392 s.
ISBN10: 0471774405, ISBN13: 9780471774402
http://pnu.edu.ru/media/filer_public/2013/04/10/6-13_segerlind_1979.pdf
9. *Зенкевич О. К. Метод конечных элементов в технике / Зенкевич О.К. – М. : Мир, 1975. – 541 с.*
Zenkevich O. K. Metod konechnyh elementov v tehnike / Zenkevich O.K. – M. : Mir, 1975. – 541 s.
ISBN10: 0070941386, ISBN13: 9780070941380
http://pnu.edu.ru/media/filer_public/2013/04/10/6-5_zenkevich_1975.pdf
10. *Сопротивление материалов. Под общ. ред. Г. С. Писаренко. – К. : Вища школа, 1979. – 696 с.*
Soprotivlenie materialov. Pod obshch. red. G. S. Pisarenko. – K. : V. shkola, 1979. – 696 s.
https://goo.gl/I3B4ZA

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